Stochastic Optimization
- Monte Carlo Methods -

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1 Announcement

Schedule

May 28th : Sample-average approximation

July 4th : Special Seminar (Brain-computer Interface)
- Time :11:00 $\sim$ 12:00 / Seminar Room - TBA
- No class on Regular class time

July 11th: Final Exam (Closed or Take home)

2 Introduction

The Final exam coverage is the overall coverage to be discussed in the class.

Coverage

- Fixed mathematical programming
- Stochastic programming- two courses model (one shot approach)
- Multi-stage programming & Dynamic Programming
- Stochastic Linear Programming with Sub problems (Branch & Cut)
- Sample Average Approximation

The overall reference of this lecture is "J. Birge and F. Louveaux, Introduction to Stochastic Programming, 2nd edition, Springer".

General stochastic model is

$$min \quad c^T + E_\xi Q(x, \xi) \tag{1}$$
$$s.t. \quad Ax = b \tag{2}$$
$$x \geq 0 \tag{3}$$

General Stochastic model (Two-Stage Program with Fixed Recourse)

$$min \quad z = c^T x + E_\xi[min \quad q(\omega)^T y(\omega)] \tag{4}$$
$$s.t. \quad Ax = b \tag{5}$$
$$T(\omega)x + Wy(\omega) = h(\omega) \tag{6}$$
$$x \geq 0, \quad y(\omega) \geq 0 \tag{7}$$

The main problem is

$$min \quad z = c^T x + Q \tag{8}$$
$$s.t. \quad Ax = b \tag{9}$$
$$x \geq 0 \tag{10}$$

Then, the sub problem is

$$min \quad q(\omega)^T y(\omega) \tag{11}$$
$$s.t. \quad T(\omega)x + Wy(\omega) = h(\omega) \tag{12}$$
$$y(\omega) \geq 0 \tag{13}$$

3 Importance Sample in the L-shape method

3.1 Introduction

$$Q^v(x) = \sum_{k=1}^{v} \frac{Q(x, \xi^k)}{v} \tag{14}$$

For a stochastic linear program is

$$min \quad c^T x + \frac{1}{v} \sum_{k=1}^{v} q_k^T y_k \tag{15}$$

$$s.t. \quad Ax = b \tag{16}$$

$$T_k x + W y_k = h_k \tag{17}$$

$$x \geq 0, \quad y_k \geq 0 \tag{18}$$

Then, a stochastic nonline program is

$$min \quad f^1(x) + \frac{1}{v} \sum_{k=1}^{v} q_k^T y_k \tag{19}$$

3.2 Sample size v and Sample Condition

how do we increase the sample size? The bigger v is the better.

3.3 L-shaped structure

Monte Carlo estimate $\xi^1, \xi^2, ..., \xi^v \sim$ the sample size v
under the given an iterate x^s

$$Q^v(x^s) = \frac{1}{v} \sum_{i=1}^{v} Q(x^s, \xi^i) \tag{20}$$

Then, we want to calculate $Q(x, \xi^i)$, instead of $Q(x^s, \xi^i)$

$$Q(x, \xi^i) \geq Q(x^s, \xi^i) + (\pi_s^i)^T (x - x^s) \quad \text{for all x} \tag{21}$$

where,

$$\nabla Q(x^s) \tag{22}$$

$$\pi_s^i \in \partial Q(x^s, \xi^i) \tag{23}$$

$$\bar{\pi}_s^v = \frac{1}{v}\sum_{i=1}^{v}\pi_s^i \tag{24}$$

$$Q^v(x^s) + (\bar{\pi}_s^v)^T(x - x^s) = LB_s^v(x) \tag{25}$$

$$Q^v(x) = \frac{1}{v}\sum_{i=1}^{v} Q(x, \xi^i) \geq LB_s^v(x) \tag{26}$$

By C.L.T, $\sqrt{v}$ X RHS $\rightarrow$ asymptotically normally distributed with a mean value

$$\sqrt{v}(Q(x^s) + \nabla Q(x^s)^T(x - x^s)) \tag{27}$$

So,

$$Q(x) \geq Q^v(x^s) + (\pi_s^v)^T(x - x^s) - \epsilon_s(x) \tag{28}$$

where $\epsilon_s(x)$ is an error term with mean 0 and Variance $\frac{1}{v}\rho^s(x)$

At iteration s, L-shaped method involves

$$min \quad c^T x + \theta \tag{29}$$

$$s.t. \quad Ax = b, \tag{30}$$

$$D_l x \geq d_l, \quad l = 1, ..., r, \tag{31}$$

$$E_l x + \theta \geq e_l, \quad l = 1, ..., s, \tag{32}$$

$$x \geq 0 \tag{33}$$

L-Shaped function is created by associating one set of the second-stage decision, y_k.

$$min \quad c^T + \sum_{k=1}^{K} p_k q_k^T y_k \tag{34}$$

$$s.t. \quad Ax = b \tag{35}$$

$$T_k x + W y_k = h_k, \quad k = 1, ..., K \tag{36}$$

$$x \geq 0, \quad y_k \geq 0, \quad k = 1, ..., K \tag{37}$$

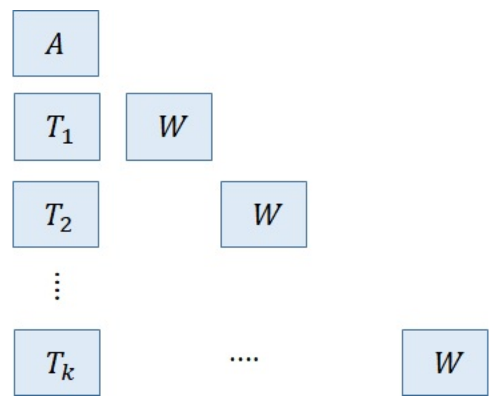

Figure 1: Block structure of the two-stage extensive form

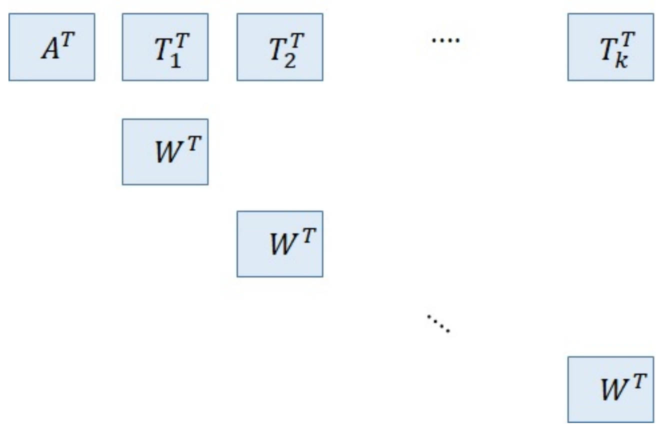

Figure 2: Block structure of the two-stage dual form

3.4 L-Shaped Algorithm

Step 0: Set $r = s = v = 0$
Step 1 : Set $v = v + 1$. Then solve the M.P.

$$min \quad z = c^T x + \theta \tag{38}$$

$$s.t. \quad Ax = b \tag{39}$$

$$D_l x \geq d_l, \quad l = 1, ..., r \tag{40}$$

$$E_l x + \theta \geq e_l, \quad l = 1, ..., s \tag{41}$$

$$x \geq 0, \quad 0 \in R \tag{42}$$

In (40), the equation is the "Cut".
Suppose that the optimal solution is (x^v, θ^v)

Step 2 : Check if $x \in K_2$. If not, add (40) and Step 1. Otherwise go to Step 3
Step 3: for k=1,...K, solve the linear program

$$min \quad w = q_k^T y \tag{43}$$

$$s.t. Wy = h_k - T_k x^v \tag{44}$$

$$y \geq 0 \tag{45}$$

Let π_k^v be the multiplier for (43)-(45) in Problem k.

$$E^{s+1} = \sum_{k=1}^{K} p_k \cdot (\pi_k^v)^T T_k \tag{46}$$

$$e_{s+1} = \sum_{k=1}^{K} p_k \cdot (\pi_k^v)^T h_k \tag{47}$$

Let $w^v = e_{s+1} - E_{s+1} x^v$. If $\theta^v \geq w^v$, the x^v is the optimal solution. Otherwise, set $s = s + 1$, add to (41) and go to Step 1.

3.5 The interpretation of L-shaped function

The master problem : (38-42). It consists of finding a proposal x, sent to the second stage. Two types of constraints are sequentially added.

Cut

- feasibility cut (40)
- Optimality cut (41)