Decomposition

Decomposition Method

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Outline

Preliminary Knowledge (I)

- $$z_j = c_B B^{-1} a_j \tag{1}$$
- $$w = c_B B^{-1} \tag{2}$$
- $$y_j = B^{-1} a_j \tag{3}$$
- $$\bar{b} = B^{-1} b \tag{4}$$
- and, Duality & KKT conditions

Preliminary Knowledge(II)

From Revised Simplex Method,

- $$\begin{bmatrix} w & C_B\hat{b} \\ B^{-1} & \hat{b} \end{bmatrix} \tag{5}$$

- Optimial Test

$$\begin{bmatrix} w & C_B\hat{b} \\ B^{-1} & \hat{b} \end{bmatrix} \begin{bmatrix} z_j - c_j \\ y_j \end{bmatrix} \tag{6}$$

- Determination of Pivotting

Example

$$Min - x_1 - 2x_2 + x_3 - x_4 - 4x_5 + 2x_6 \tag{7}$$

$$s.t. \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 6 \tag{8}$$

$$2x_1 - x_2 + 2x_3 + x_4 \leq 4 \tag{9}$$

$$x_3 + x_4 + 2x_5 + x_6 \geq 0 \tag{10}$$

let B $= a_7, a_8, a_9$

Example, con'td

$$w = c_B B^{-1} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \tag{11}$$

$$C_B \hat{b} = 0 \tag{12}$$

$$B_{-1} = I \tag{13}$$

$$\hat{b} = \begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix} \tag{14}$$

$$z_j - c_j = \begin{bmatrix} 1 & 2 & -1 & 1 & 4 & -2 \end{bmatrix} \tag{15}$$

so, x_5 is selected

$$y_5 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \tag{16}$$

example,cont'd

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} 0 & 0 & -2 & -8 \\ 1 & 0 & -1/2 & 4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1/2 & 2 \end{bmatrix} \tag{18}$$

then, cont'd

Concept

Column Generation

- one part $\Rightarrow$ General (Complicating) Structure + Special (Efficient) Structure
- Master prblem + Sub problem
- Master problem $\rightarrow$ Sub problem : Cost Coefficients
- Sub problem $\rightarrow$ Master : Columns

Decomposition Algorithm

Initial M.P.

$$Min \quad cx \tag{19}$$

$$s.t. \quad Ax = b \tag{20}$$

$$x \in X \tag{21}$$

- using Convex combination of X

$$x = \sum_{j=1}^{t} \lambda_j x_j \quad , \quad \sum_{j=1}^{t} \lambda_j = 1 \quad , \quad \lambda_j \geq 0 \tag{22}$$

Decomposition,cont'd : Master Problem

- Variable : λ_j

$$Min \quad \sum_{j=1}^{t}(cx_j)\lambda_j \tag{23}$$

-

$$s.t. \quad \sum_{j=1}^{t}(Ax_j)\lambda_j = b \tag{24}$$

-

$$\sum_{j=1}^{t}\lambda_j = 1 \tag{25}$$

-

$$\lambda_j \geq 0 \tag{26}$$

- t is large $\rightarrow$ Large Scale Programming

Decomposition, Sub problem

- Introducing Dual variable w and α

$$z_k - c_k = max \quad z_j - \hat{c}_j = max \quad (w, \alpha) \begin{bmatrix} Ax_j \\ 1 \end{bmatrix} - cx_j \tag{27}$$

- $$max \quad wAx_j + \alpha - cx_j \tag{28}$$

- $$max \quad (wA - c)x_j + \alpha = max \quad (wA - c)x + \alpha \tag{29}$$

Sub Problem

- $$Max(wA - c)x + \alpha \tag{30}$$
- $$s.t. \quad x \in X \tag{31}$$

Preliminary Terms

- $$\hat{c} = cx_j \tag{32}$$
- $$(w, \alpha) = \hat{c}_B B^{-1} \tag{33}$$
- $$\bar{b} = B^{-1} \begin{bmatrix} b \\ 1 \end{bmatrix} \tag{34}$$

Main algorithm - Decomposition (I)

- Step 1 : An initial feasible solution
- Step 2 : Find w, α, $\bar{b}$
- Step 3 : Solve Sub problem

$$Max \quad (wA - c)x + \alpha \tag{35}$$

$$s.t. \quad x \in X \tag{36}$$

- Step 4 : Optimal test from Step 3 Formulas

Optimality Test

if x_k's $\quad z_k - \hat{c}_k = 0 \rightarrow$ Optimal!

Decomposition (II)

- Step 5 (Not optimial from Step 4) :

From Bazaraa, Jarvis, Sherali

$$Min \quad -2x_1 - x_2 - x_3 + x_4 \tag{37}$$

$$x_1 + x_3 \leq 2 \tag{38}$$

$$x_1 + x_2 + 2x_4 \leq 3 \tag{39}$$

$$x_1 \leq 2 \tag{40}$$

$$x_1 + 2x_2 \leq 5 \tag{41}$$

$$-x_3 + x_4 \leq 2 \tag{42}$$

$$2x_3 + x_4 \leq 6 \tag{43}$$

$$x_i \geq 0 \tag{44}$$

Broader region

$$x = \sum_{j=1}^{t} \lambda_j x_j + \sum_{j=1}^{l} u_j d_j \tag{45}$$

$$\sum_{j=1}^{t} \lambda_j = 1, \lambda_j \geq 0, \mu \geq 0 \tag{46}$$

Broader region, Cont'd

Then, the problem is modeled to

$$Min \quad \sum_{j=1}^{t}(cx_j)\lambda_j + \sum_{j=1}^{l}\mu_j d_j \tag{47}$$

$$s.t. \quad \sum_{j=1}^{t}(AX_j)\lambda_j + \sum_{j=1}^{l}(Ad_j)\mu_j = b \tag{48}$$

$$\sum_{j=1}^{t}\lambda_j = 1, \lambda_j \geq 0, \mu \geq 0 \tag{49}$$

Optimality test

λ_j is non-basic

$$0 \geq z_j - \hat{c}_j = \begin{bmatrix} w & \alpha \end{bmatrix} \begin{bmatrix} Ax_j \\ 1 \end{bmatrix} - cx_j = wAx_j + \alpha - cx_j \quad (50)$$

μ_j is non-basic

$$0 \geq z_j - \hat{c}_j = \begin{bmatrix} w & \alpha \end{bmatrix} \begin{bmatrix} Ad_j \\ 0 \end{bmatrix} - cx_j = wAx_j - cd_j \quad (51)$$

Master & Subproblem

Master Problem

$$\begin{bmatrix} (w,\alpha) & \hat{c}\bar{b} \\ B^{-1} & \bar{b} \end{bmatrix} \tag{52}$$

Sub Problem

$$Max \quad (wA - c)x + \alpha \tag{53}$$

$$s.t. \quad x \in X \tag{54}$$

Broader example

$$min \quad -x_1 - 2x_2 - x_3 \tag{55}$$

$$s.t. \quad x_1 + x_2 + x_3 \leq 12 \tag{56}$$

$$-x_1 + x_2 \leq 2 \tag{57}$$

$$-x_1 + 2x_2 \leq 8 \tag{58}$$

$$x_3 \leq 3 \tag{59}$$

Block Diagonal Structure

$$Min \quad c_1x_1 + c_2x_2 + ... + c_Tx_T \tag{60}$$

$$s.t. \quad A_1x_1 + A_2x_2 + ... + A_Tx_T = b \tag{61}$$

$$B_1x_1 \qquad \leq b_1 \tag{62}$$

$$B_2x_2 \qquad \leq b_2 \tag{63}$$

...

$$B_Tx_T \leq b_n \tag{64}$$

$$x_i \geq 0 \tag{65}$$

Angular Structure

Block Diagonal Structure or Angular Structure

$$x_i = \sum_{j=1}^{t_i} \lambda_{ij} x_{ij} + \sum_{j=1}^{l_i} \mu_{ij} d_{ij} \quad (66)$$

$$\sum_{j=1}^{t_i} \lambda_{ij} = 1 \quad (67)$$

$$\lambda_{ij} \geq 0, \mu_{ij} \geq 0 \quad (68)$$

Conversion using Angular Structure

$$Min \quad \sum_{i=1}^{T}\sum_{j=1}^{t_i}(c_i x_{ij})\lambda_{ij} + \sum_{i=1}^{T}\sum_{j=1}^{l_i}(c_i d_{ij})\mu_{ij} \tag{69}$$

$$s.t. \quad \sum_{i=1}^{T}\sum_{j=1}^{t_i}(A_i x_{ij})\lambda_{ij} + \sum_{i=1}^{T}\sum_{j=1}^{l_i}(A_i d_{ij})\mu_{ij} = b \tag{70}$$

$$\sum_{j=1}^{t_i}\lambda_{ij} = 1 \tag{71}$$

$$\lambda_{ij} \geq 0, \mu_{ij} \geq 0 \tag{72}$$