# <March 28> → Quantum operator

1. Super-position

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

$|u_1\rangle$ $|u_2\rangle$

basis basis/span

$\langle u_1 | u_2 \rangle$ ⇒ inner product.

$|a\rangle$
$|b\rangle$

$\langle a|b\rangle = \langle b|a\rangle$

$|a\rangle = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2|0\rangle + 3|1\rangle$

$|b\rangle = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \Rightarrow 5|0\rangle + 6|1\rangle$

$\| a \| = $

$|\tilde{a}\rangle$

$|a\rangle = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Cat vector

$\langle a| = (2, 3)$

bravector.

$|b\rangle = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

$\langle b| = (5, 6)$

* Inner product

$\langle a|b\rangle = \langle a| \cdot |b\rangle$

$(2,3) \cdot \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

$= 28$.

$\langle b|a\rangle = \langle b| \cdot |a\rangle$

$= (5,6) \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$= 28$.

$|\psi\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix}$ , $|\phi\rangle = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\rightarrow \langle\phi| = (2, 3)$

$\langle\psi| = (1, -i)$

$\langle\psi|\phi\rangle = 2-3i$

$\neq$

$\langle\phi|\psi\rangle = 2+3i$

$|\psi\rangle^{*} \rightarrow$ conjuge

$(|\psi\rangle)^{+} \rightarrow$ hermitian adjoint

$= (|\psi\rangle^{*})^{T}$

* in Superposition.

$\phi = \frac{1}{\sqrt{5}}|0\rangle + \sqrt{\frac{4}{5}}|0\rangle$

$|\tilde{\psi}\rangle = \underline{\frac{1}{\sqrt{3}}|0\rangle} + \sqrt{\frac{2}{3}}\,\underline{|1\rangle}$

$\underline{\langle\tilde{\psi}|\tilde{\psi}\rangle} = \left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)\begin{pmatrix}\frac{1}{\sqrt{3}}\\ \sqrt{\frac{2}{3}}\end{pmatrix} = \underline{1}.$

$\langle 0|\psi\rangle^2 = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$

$\langle 1|\psi\rangle^2 = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{2}{3}$

2. bra vector ( dual vector )

$$|u\rangle \rightsquigarrow \langle u| = (|u\rangle)^{\dagger}$$ → Hermitian

adjoint

‖

conjugate

$$|\phi\rangle = \boxed{\frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle}$$

$$\langle\phi| = \left(\frac{-i}{2}, \frac{\sqrt{3}}{2}\right)$$

3. Norm ⇒

$$\sqrt{\langle a|a\rangle}$$

$$|v\rangle = \begin{pmatrix} -1 \\ 3i \\ i \end{pmatrix}$$

$$\|v\| = \sqrt{\langle v | v \rangle} = \sqrt{(-1, -3i, -i)\begin{pmatrix} -1 \\ 3i \\ i \end{pmatrix}} = \sqrt{1+9+\ldots}$$

$$= \sqrt{11}$$

$$* \langle \star | \psi \rangle$$

$\star \to$ detector (detecting operator)

$\psi \to$ states

e.g.

$$|\tilde{\psi}\rangle = \frac{1}{\sqrt{5}}|u_1\rangle - i\sqrt{\frac{7}{\sqrt{5}}}|u_2\rangle + \frac{1}{\sqrt{3}}|u_3\rangle$$

$$\boxed{\langle u_1 | \psi \rangle}^2 = \left(\langle u_1 \middle| \frac{1}{\sqrt{5}}|u_1\rangle - i\sqrt{\frac{7}{5}}|u_2\rangle + \frac{1}{\sqrt{3}}|u_3\rangle\right)^2$$

$$= \left(\frac{1}{\sqrt{5}}\right)^2 = \frac{1}{5}$$

* operator $\Rightarrow$ $\hat{A}$ $\rightsquigarrow$ caret

$$\boxed{\hat{A}\cdot|\psi\rangle} = \hat{A}\left(\alpha|\psi_0\rangle + \beta|\psi_1\rangle\right)$$

$$|\psi\rangle = \alpha|\psi_0\rangle + \beta|\psi_1\rangle$$

$$= \alpha\hat{A}|\psi_0\rangle + \beta\hat{A}|\psi_1\rangle$$

$$\boxed{= \sum_{i=1}^{n} \alpha_i \left(\hat{A}|\psi_i\rangle\right)}$$

* Pauli matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \frac{? + ?}{?}$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \Rightarrow$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow |0\rangle\langle 0| - |1\rangle\langle 1|$$

* outer product $\left\{ \begin{array}{l} |\phi\rangle \\ |\phi\rangle \end{array} \right.$

$|\phi\rangle \cdot \langle\phi|$

---

a state $|\chi\rangle$

$$|\phi\rangle \cdot \langle\phi| \cdot |\chi\rangle = \boxed{|\phi\rangle \cdot \langle\phi|\chi\rangle}$$

* example

$$\begin{cases} \hat{A} = |0\rangle\cdot\langle 0| - |1\rangle\cdot\langle 1| \\ |\phi\rangle = \alpha|0\rangle + \beta|1\rangle \end{cases}$$

$$\hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{A}\cdot|\phi\rangle = \left(|0\rangle\cdot\langle 0| - |1\rangle\cdot\langle 1|\right)\left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$\alpha(|0\rangle) - \beta|1\rangle)$$

$$\boxed{|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \qquad |\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix}}$$

$$|\psi\rangle \cdot \langle\phi| = \begin{pmatrix} a \\ b \end{pmatrix} \cdot (c^*, d^*) = \begin{pmatrix} ac^* & ad^* \\ bc^* & bd^* \end{pmatrix}$$

$$\langle\phi| = (c^*, d^*)$$

$$\neq$$

$$|\phi\rangle \cdot \langle\psi| = \begin{pmatrix} c \\ d \end{pmatrix} (a^* \; b^*) = \begin{pmatrix} ca^* & cb^* \\ da^* & db^* \end{pmatrix}$$

# * Span / basis

$|u_i\rangle \rightsquigarrow$ Span iff

$$\boxed{\sum_{i=1}^{n} |u_i\rangle\langle u_i| = \hat{I}}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

//

$$|u_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|u_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$|u_1\rangle\langle u_1| + |u_2\rangle$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}(1, 0)$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$|\psi\rangle = \hat{I}|\psi\rangle = \left(\sum_i |u_i\rangle\langle u_i|\right)|\psi\rangle$$

$$= \sum_i |u_i\rangle \underbrace{\langle u_i|\psi\rangle}_{c_i}$$

$$= \sum_i c_i |u_i\rangle$$

* Dimension generalization

$$\hat{A} = \underline{\hat{I} \cdot \hat{A} \cdot \hat{I}} = \left(\sum_i |u_i\rangle\langle u_i|\right) \hat{A} \left(\sum_j |u_j\rangle\langle u_j|\right)$$

$$= \sum_j \sum_i |u_i\rangle \cdot \langle u_i|\hat{A}|u_j\rangle \cdot \langle u_j|$$

Constant.

$$\hat{A} = \begin{pmatrix} \langle u_1|\hat{A}|u_1\rangle & \cdots & \langle u_1|\hat{A}|u_n\rangle \\ \vdots & & \vdots \\ \langle u_n|\hat{A}|u_1\rangle & & \langle u_n|\hat{A}|u_n\rangle \end{pmatrix} = \sum_j \sum_i \langle u_i|\hat{A}|u_j\rangle \cdot |u_i\rangle\langle u_j|$$

$$\begin{cases} |\psi\rangle = a|1\rangle + b|2\rangle + c|3\rangle \\ |\phi\rangle = e|1\rangle + f|2\rangle + g|3\rangle \end{cases}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} e^* & f^* & g^* \end{pmatrix}$$

$$\therefore |\psi\rangle\langle\phi| = \begin{pmatrix} ae^* & af^* & ag^* \\ be^* & bf^* & bg^* \\ ce^* & cf^* & cg^* \end{pmatrix}$$

Let $\hat{A} \to ?$

$$\begin{cases} \hat{A}|0\rangle = |0\rangle \\ \hat{A}|1\rangle = -|1\rangle \end{cases}$$

$$A = \begin{bmatrix} \langle 0|A|0\rangle & \langle 0|A|1\rangle \\ \langle 1|A|0\rangle & \langle 1|A|1\rangle \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

$\langle 0|A|0\rangle = 0$, $\langle 0|A|1\rangle = -1$, $\langle 1|A|0\rangle = 0$, $\langle 1|A|1\rangle = -1$

$A|0\rangle = |1\rangle$

$A|1\rangle = |0\rangle$

$$A = \begin{bmatrix} \langle 0|A|0\rangle & \langle 0|A|1\rangle \\ \langle 1|A|0\rangle & \langle 1|A|1\rangle \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$$

$$\begin{bmatrix} \overset{1}{\langle 0|A|0\rangle} & \overset{0}{\langle 0|A|1\rangle} \\ \underset{1}{\langle 1|A|0\rangle} & \underset{0}{\langle 1|A|1\rangle} \end{bmatrix}$$

$$\begin{cases} A|0\rangle = i|1\rangle \\ A|1\rangle = -i|0\rangle \end{cases}$$

$$A \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \underline{\underline{Y}}$$

* Quantum Axiom

$$|\psi_0\rangle \xrightarrow{\text{"}U\text{"}} |\psi_t\rangle$$

unitary operator

$m_1$: $p_1 = |\langle \cdot|\psi_t\rangle|^2$

$m_2$: $p_2 = |\langle \cdot|\psi_t\rangle|^2$

$(\langle\psi_0|m_1 U$

$$|\psi_t\rangle = |U\rangle\langle U|$$

$$\langle\psi_0|U|\psi_0\rangle = \text{constant}$$

* Hermitian operator
  Unitary operator

$$|\psi_0\rangle \xrightarrow{A} |\psi_t\rangle$$

$$|\psi_t\rangle = A|\psi_0\rangle$$

$$\boxed{|\psi_0\rangle = A^{-1}|\psi_t\rangle}$$

d!

$A \sim$ unitary operat

* Hermetian operator

" $\hat{A}$ "

$\hat{A}^{\dagger}$ Hermetian adjoint.

$$\langle a | \hat{A}^{\dagger} | b \rangle = \langle b | \hat{A} | a \rangle$$

$$(\alpha A)^\dagger = \alpha^* \cdot A^\dagger.$$

$$(|\psi\rangle)^\dagger = \langle\psi|$$

$$(\langle\psi|)^\dagger = |\psi\rangle$$

$$(\hat{A}\cdot\hat{B})^\dagger = \hat{B}^\dagger \cdot \hat{A}^\dagger.$$

$$(\hat{A}|\psi\rangle)^\dagger = |\psi\rangle^\dagger \cdot \hat{A}^\dagger$$

$$= \langle\psi|\cdot\hat{A}^\dagger.$$

$$(AB|\psi\rangle)^\dagger = \langle\psi|\cdot B^\dagger,$$

$$\hat{A} = 2\,|0\rangle\langle 1| - i\,|1\rangle \cdot \langle 0|$$

$$\hat{A}^\dagger = \left( \quad , \quad \right)^\dagger$$

$$2\,(\langle 1|)^\dagger \cdot (|0\rangle)^\dagger + i\,(\langle 0|)^\dagger (|1\rangle)^\dagger$$

$$= 2\,|1\rangle \cdot \langle 0| + i\,|0\rangle \cdot \langle 1|$$

$$\hat{A} = \hat{A}^\dagger$$

$$\Downarrow$$

$\hat{A}$ ~ Hermitian operator

$$\begin{cases} A = A^\dagger \\ A \cdot A^\dagger = I \end{cases}$$

$A$ $\Downarrow$ unitary operator.

Every Pauli matrix

$\Downarrow$

~~H & U~~

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\Downarrow$

$$X^\dagger = \begin{pmatrix} & ? & \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

* Relationship betwee eigenVector / eigenValue.

$$\boxed{A \cdot |\psi_0\rangle = \lambda \cdot |\psi_0\rangle}$$

operator

$|\psi\rangle = \alpha|0\rangle + \beta$

$$|\psi_t\rangle = \underline{\lambda|\psi_0\rangle}$$

$$A|\psi\rangle = \underbrace{\lambda}_{\text{eigen value of } A} \boxed{|\psi\rangle} \Rightarrow \text{eigen vector of } A$$

$$\det|A - \lambda \cdot I| = 0$$

$$A = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \Rightarrow \left|\begin{pmatrix} 2-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix}\right| = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}.$$

$$\boxed{(2-\lambda)\cdot(-1-\lambda) - (-1)\cdot(1) = 0}$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$$

phase operator $\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$

$(1-\lambda)(e^{i\frac{\pi}{4}}-\lambda)=0$

$$\boxed{\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= e^{i\frac{\pi}{4}} \end{aligned}}$$

$A\,|\psi\rangle = \lambda\,|\psi\rangle$

$\begin{pmatrix} a \\ b \end{pmatrix}$

eigen value

$$\rightarrow \left| \begin{pmatrix} 1-\lambda & 0 \\ 0 & e^{i\frac{\pi}{4}}-\lambda \end{pmatrix} \right| = 0$$

Case $\lambda_1 = 1 \Rightarrow$

$$T \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 1 \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} a \\ e^{i\frac{\pi}{4}} \cdot b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \begin{pmatrix} a = 1 \\ b = 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Case $\lambda_2 = e^{i\frac{\pi}{4}}$

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = e^{i\frac{\pi}{4}} \begin{pmatrix} a \\ b \end{pmatrix}$$

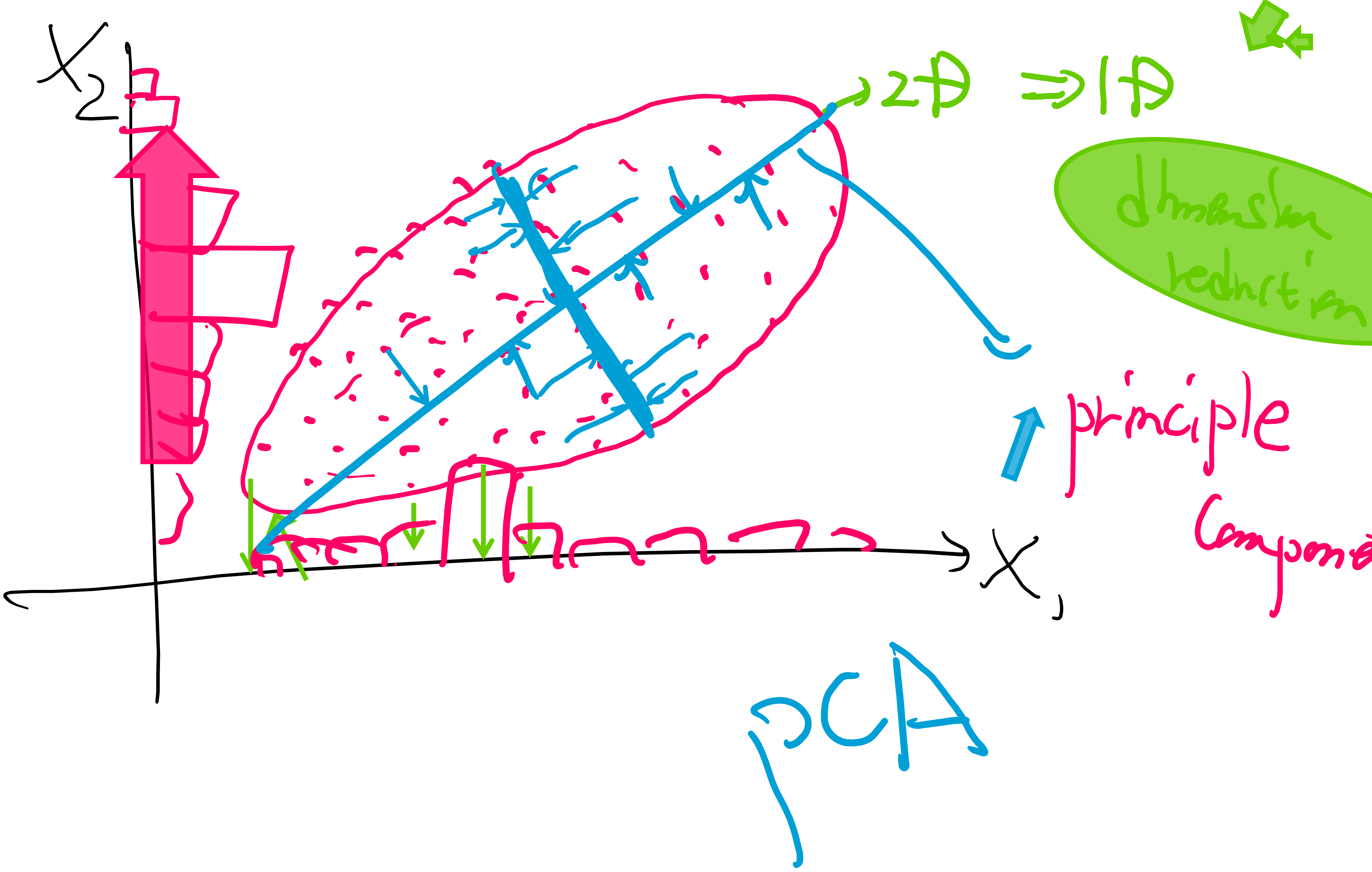
$$\begin{pmatrix} a = 0 \\ b = \underline{1} \end{pmatrix}$$

$$T = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{i\frac{\pi}{4}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

* eigen vector

① Quantum ⇒ "A" ~> Qubit ( eigen Va
eigen Vect

② Data



$X_2$
$X_1$
2D ⇒ 1D
dimension reduction
principle component
PCA

Hermite operator has eigenvector

orthogonal

$$\langle \psi_i | \psi_j \rangle = 0$$

## * Specturm analysis

$$A = \sum_{i=1}^{m} a_i \boxed{|u_i\rangle\langle u_i|}$$

eigenvalue

eigenvector

onter product

$$A = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix} = -|u_1\rangle\langle u_1| + |u_2\rangle\langle u_2| + |u_3\rangle\langle u_3|$$

eigen Vectors

$$\lambda_1 = -1 \Longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = |u_1\rangle$$

$$\lambda_2 = 1 \Longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} = |u_2\rangle$$

$$\lambda_3 = 1 \Longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |u_3\rangle$$