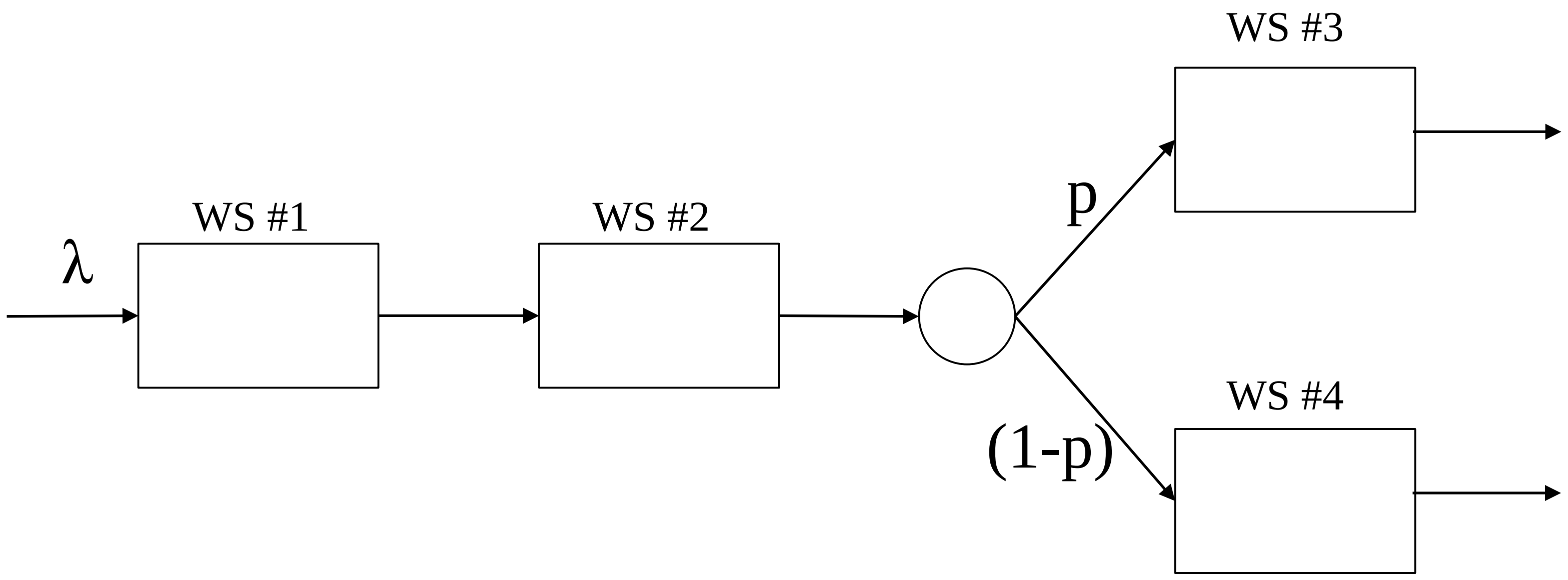
Multi-Stage Models



HYUNSOO LEE

Review (1)

- Stages
  - Process
  - Simulation
  - Control

Review (2)

- M/G/C

$$CT_q = \left(\frac{U}{1-U}\right) \cdot E[S] \cdot \left(\frac{1+C^2[S]}{2}\right) \cdot \left(\frac{U^{\sqrt{2C+2}-2}}{C}\right)$$

- G/G/C

$$CT_q = \left(\frac{U}{1-U}\right) \cdot E[S] \cdot \left(\frac{C^2[A]+C^2[S]}{2}\right) \cdot \left(\frac{U^{\sqrt{2C+2}-2}}{C}\right)$$

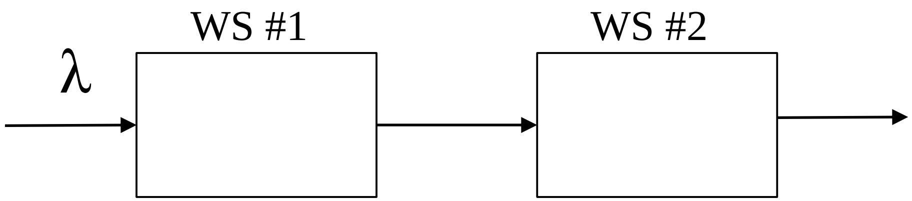
Contents

- Multi-stage Models
  - Multi-stage Serial Processes
  - Multi-stage Merging Processes
  - Multi-stage Split Processes

  - Application

Serial Processes (1)

- Idea from two-stage processes



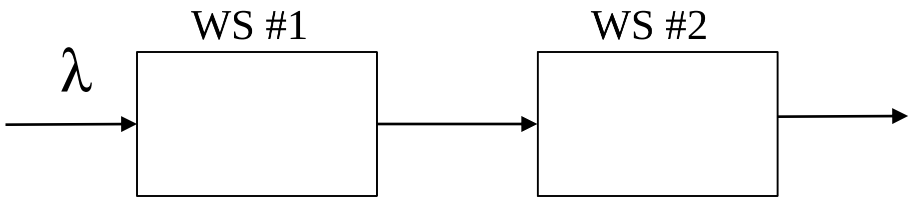
$$CT_{Total} \neq CT_1 + CT_2$$

$$CT_{Total} = CT_1 + CT_2'$$

$$CT_2' = \left(\frac{U}{1-U}\right) \cdot E[S] \cdot \left(\frac{C^2[A] + C^2[S]}{2}\right) \cdot \left(\frac{U^{\sqrt{2C+2}-2}}{C}\right) + E[S]$$

Serial Processes (2)

- Calculation of $C^2[A]$



$$C^2[A] = C_a^2(2) = C_d^2(1)$$

– Buzzacott and Shanthikumar (1963)

$$C_d^2(1) \approx (1 - U^2) \cdot C_a^2(1) + U^2 \cdot C_s^2$$

Serial Processes (3)

- Estimation of $C_d^2(1)$
  - G/G/1

$$C_d^2(1) \approx (1-U^2)\cdot C_a^2(1) + U^2 \cdot C_s^2$$

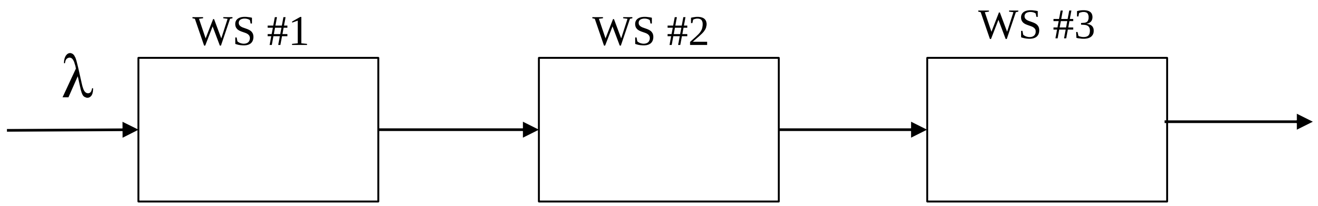
  - G/G/C

$$C_d^2(1) \approx (1-U^2)\cdot C_a^2(1) + U^2 \cdot \frac{C_s^2 + \sqrt{c} - 1}{\sqrt{c}}$$

Where, $U = \dfrac{\lambda}{c \cdot \mu}$

Serial Processes (4)

- Expansion to general processes
  - Given Data



$C_a^2(1)$ $E[S_1]$ $E[S_2]$ $E[S_3]$

$C_s^2(1)$ $C_s^2(2)$ $C_s^2(3)$

$$WIP_{Sys} = ?$$

$$CT_{Sys} = CT_S(1) + CT_S(2) + CT_S(3)$$

Serial Processes (5)

- Cont'd, #1 machine

$$CT_{Sys} = CT_S(1) + CT_S(2) + CT_S(3)$$

$$CT_s(1) = \left(\frac{U_1}{1-U_1}\right) \cdot E[S] \cdot \left(\frac{C_a^2(1) + C_s^2(1)}{2}\right) \cdot \left(\frac{U_1^{\sqrt{2C+2}-2}}{C}\right) + E[S_1]$$

Where, $U_1 = \lambda_1 \cdot E[S_1]$

Serial Processes (6)

- Cont'd, #2 machine

$$CT_{Sys} = CT_S(1) + CT_S(2) + CT_S(3)$$

$$CT_s(2) = \left(\frac{U_2}{1-U_2}\right) \cdot E[S_2] \cdot \left(\frac{C_a^2(2) + C_s^2(2)}{2}\right) \cdot \left(\frac{U_2^{\sqrt{2C+2}-2}}{C}\right) + E[S_2]$$

$$C_a^2(2) = C_d^2(1) \approx (1-U_1^{\,2}) \cdot C_a^2(1) + U_1^{\,2} \cdot \frac{C_s^2(1) + \sqrt{c} - 1}{\sqrt{c}}$$

Where, $U_1 = \lambda_1 \cdot E[S_1]$

$U_2 = \lambda_2 \cdot E[S_2]$

Serial Processes (7)

- Cont'd, #3 machine

$$CT_{Sys} = CT_S(1) + CT_S(2) + CT_S(3)$$

$$CT_s(3) = \left(\frac{U_3}{1-U_3}\right) \cdot E[S_3] \cdot \left(\frac{C_a^2(3) + C_s^2(3)}{2}\right) \cdot \left(\frac{U_3^{\sqrt{2C+2}-2}}{C}\right) + E[S_3]$$

$$C_a^2(3) = C_d^2(2) \approx (1-U_2^{\ 2}) \cdot C_a^2(2) + U_2^{\ 2} \cdot \frac{C_s^2(2) + \sqrt{c} - 1}{\sqrt{c}}$$

Where, $U_2 = \lambda_2 \cdot E[S_2]$

$U_3 = \lambda_3 \cdot E[S_3]$

Serial Processes (8)

- Finally,

$$CT_{Sys} = CT_S(1) + CT_S(2) + CT_S(3)$$

$$WIP_{Sys} = \lambda_1 \cdot CT_S(1)$$

Homework #1

- Consider a three-workstation factory with serial flow

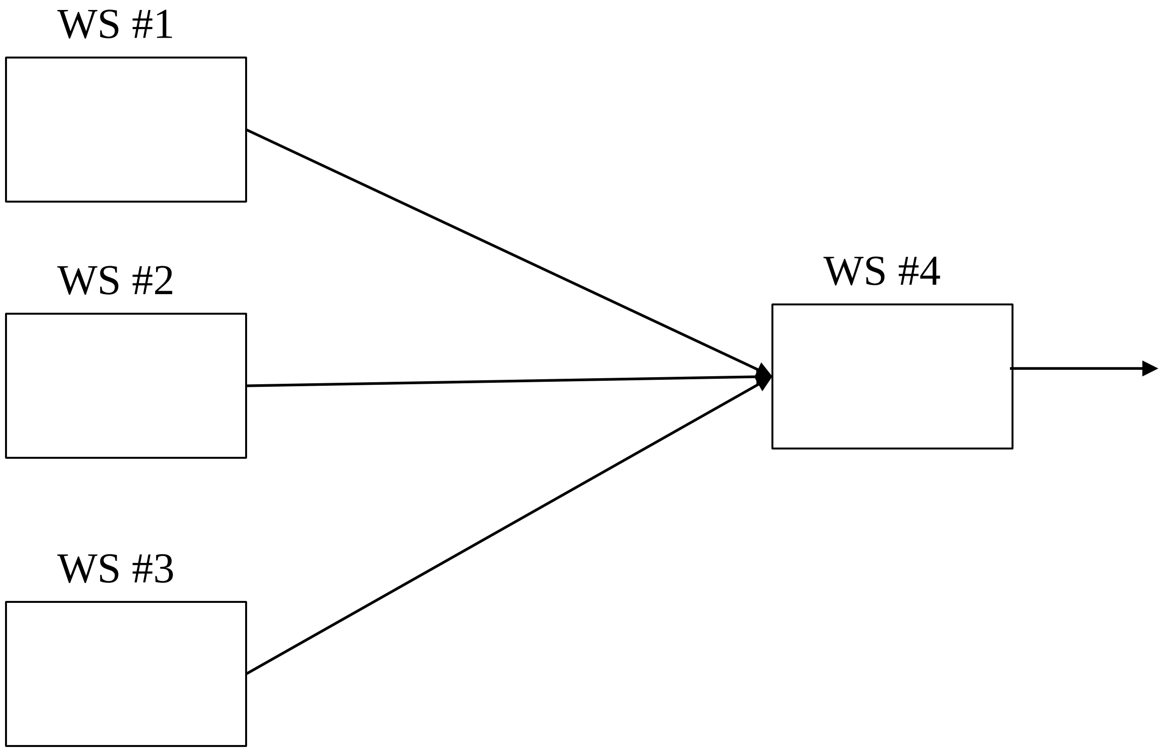
Workstation	$E[S_i]$	C_s^2
1	12 min	2.0
2	9 min	0.7
3	13.2 min	1.0

  - A mean of 15 min (4 jobs per hour)
  - A Squared coefficient of variation = 0.75

$$TH_{Sys}$$
$$CT_{Sys}$$
$$WIP_{Sys}$$

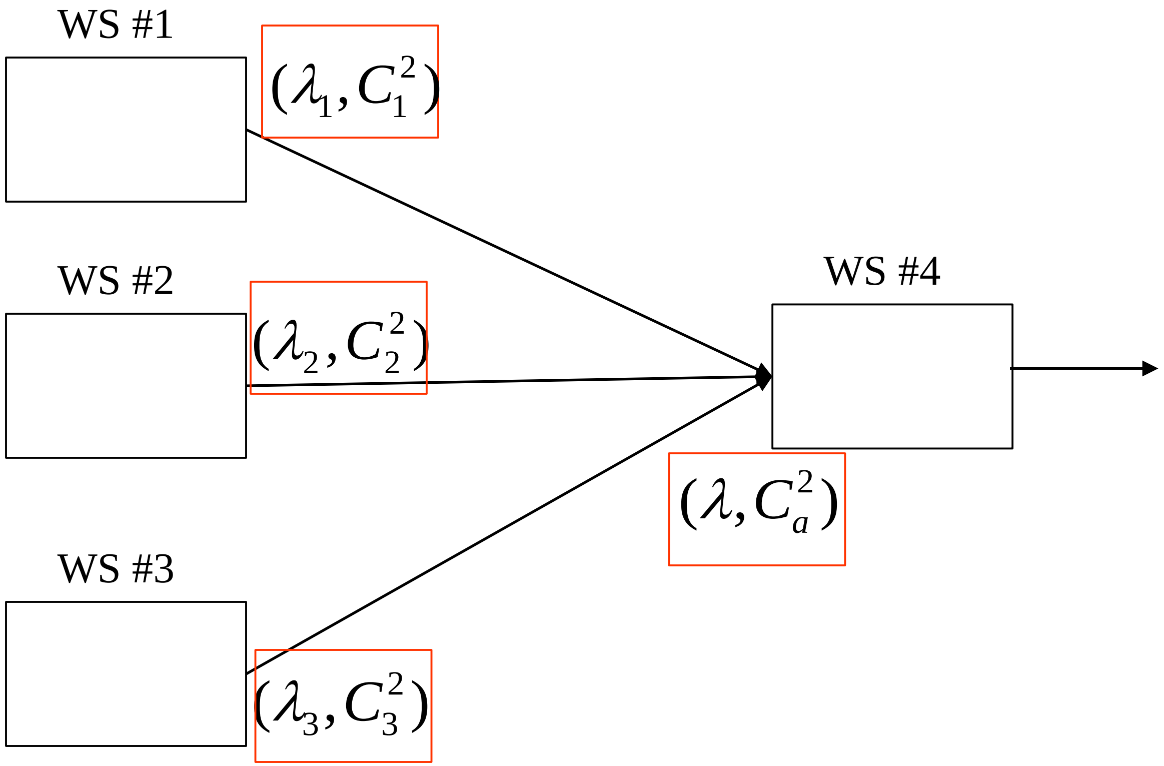
Merging Processes (1)

- Scenario



Merging Processes (2)

- Scenario, Cont'd



Merging Processes (3)

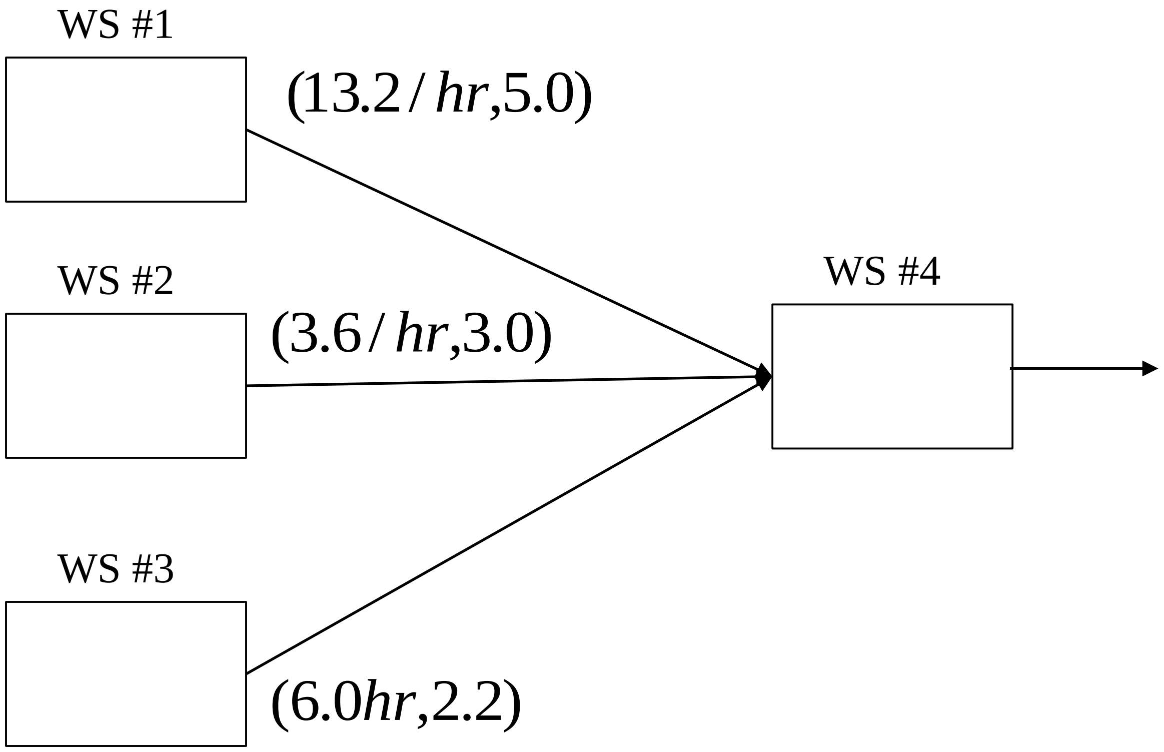
- Conclusion

$$\lambda = \sum_{i=1}^{n} \lambda_i$$

$$C_a^2 = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} \cdot C_i^2$$

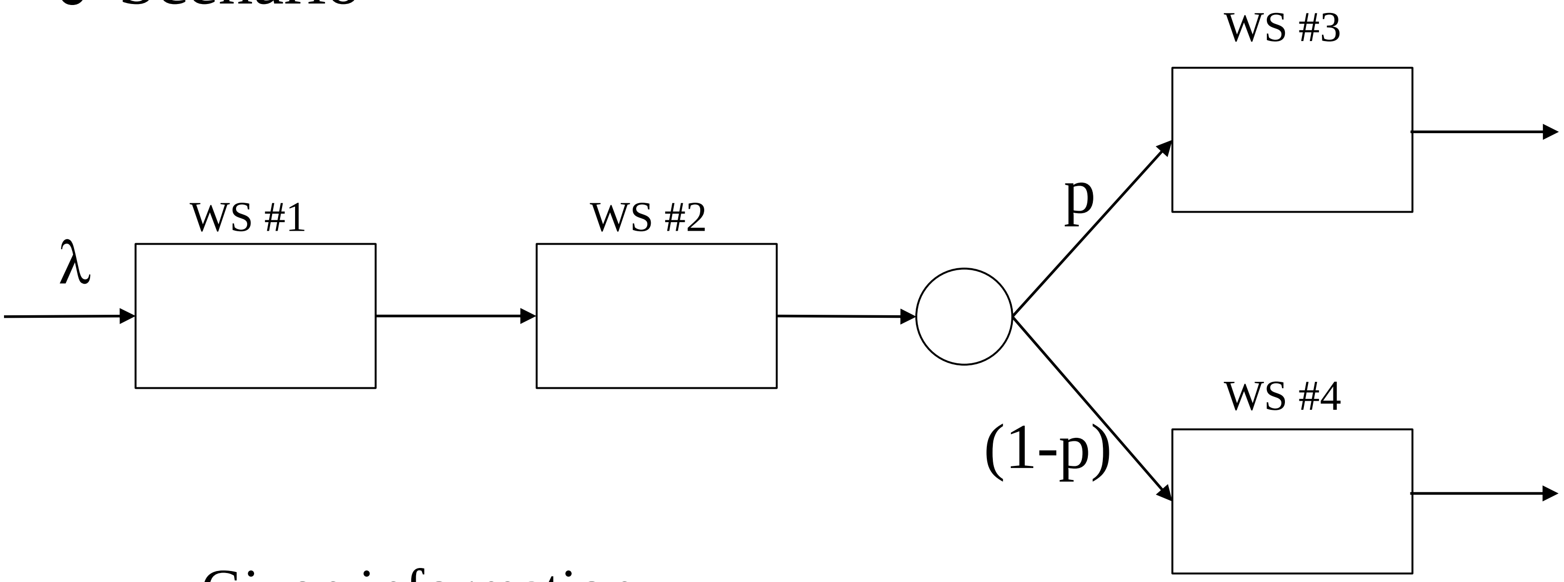
Homework #2

- Calculate V_a



Split Processes (1)

- Scenario



  - Given information

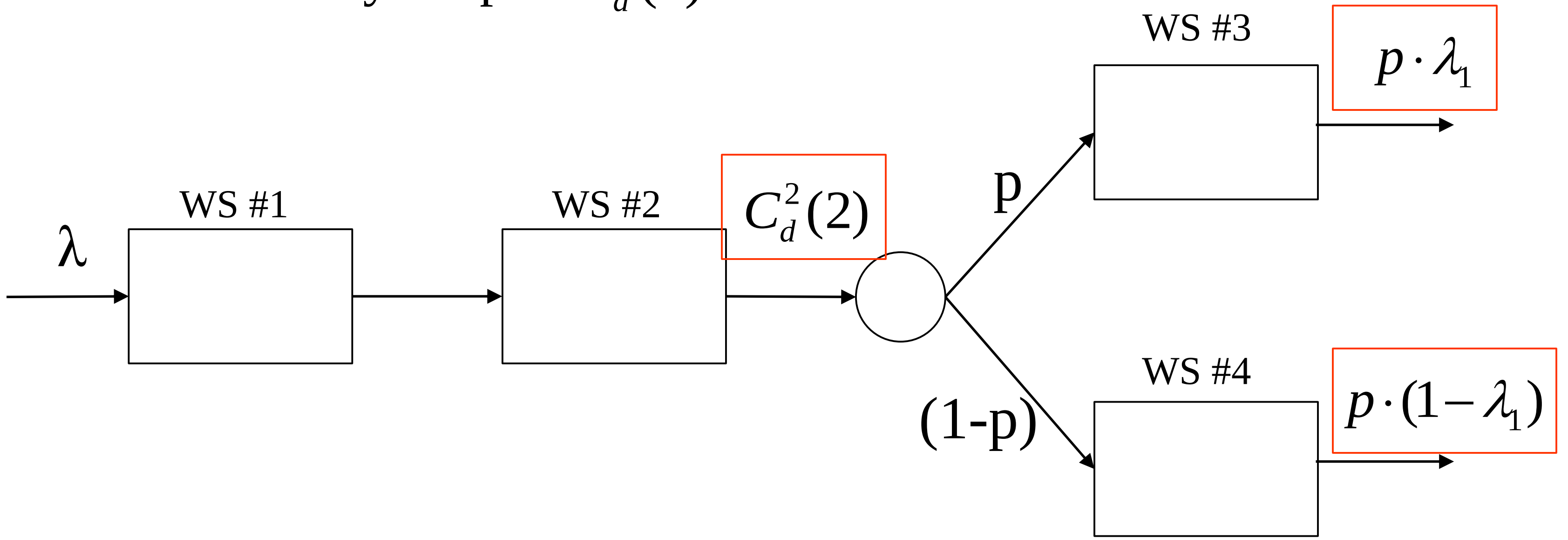
$$E[S_i], _{i=1,2,3,4}$$

$$C_s^2(i), _{i=1,2,3,4}$$

$$p, _{0 \le p \le 1}$$

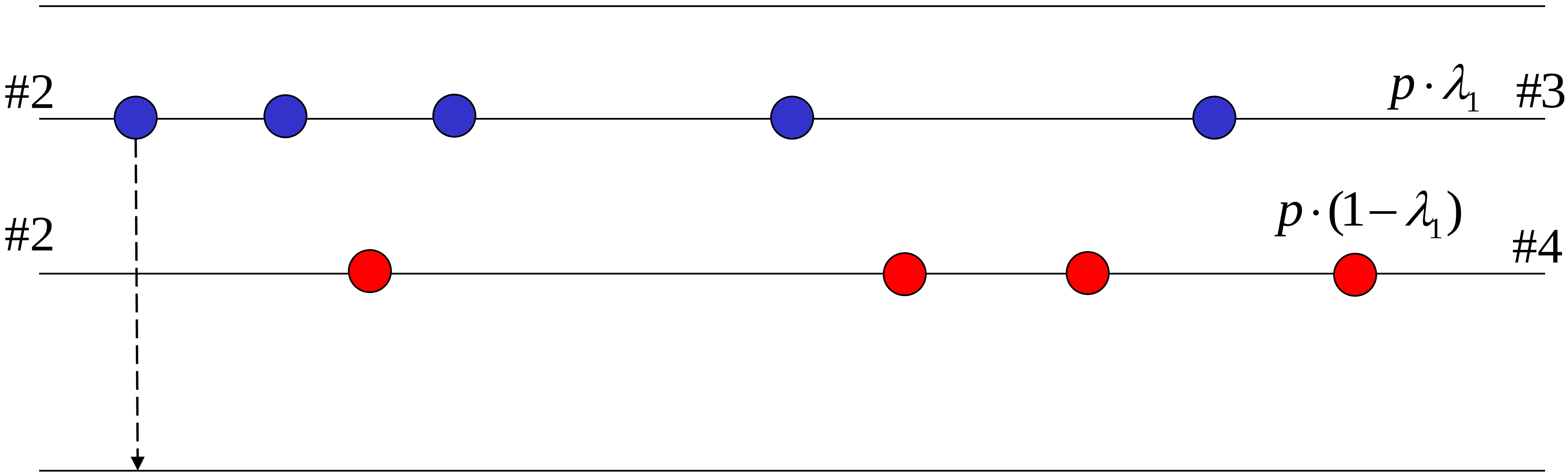
Split Processes (2)

- Scenario, cont'd
  - Type 1 : 1→2→3
  - Type2 : 1→2→4
  - How do you split $C_d^2(2)$?



Split Processes (3)

- Analysis of $C_d^2(2)$



$$C_d^2(2) \approx (1-U_2{}^2) \cdot C_a^2(2) + U_2{}^2 \cdot \frac{C_s^2(2)+\sqrt{c}-1}{\sqrt{c}}$$

Split Processes (4)

- Analysis of Stage #4

$$\#1(4) = T_1 + T_2 + T_3$$

$$\#2(4) = T_4 + T_5 + T_6$$

$$\#3(4) = T_7$$

$$\#4(4) = T_8 + T_9$$

$$T_4 = T_1 + \cdots + T_N = \sum_{i=1}^{N} T_i$$

$$E[T_4] = E\left[\sum_{i=1}^{N} T_i\right] = E[T] \cdot \boxed{E[N]}$$

Split Processes (5)

- Analysis of Stage #4, Cont'd

$$E[N]$$

$$P\{N=n\}=(1-p)^{n-1}\cdot p, n=1,2,\cdots$$

$$E[N]=\frac{1}{p}$$

$$V[N]=\frac{1-p}{p^2}$$

$$CoV[N]=1-p$$

$$E[T_4]=\frac{E[T]}{P}$$

$$V[T_4]=\frac{V[T]}{P}+\frac{(1-p)E(T)^2}{P^2}$$

Split Processes (6)

- Generalization

$$E[T_4] = \frac{E[T]}{P}$$

$$V[T_4] = \frac{V[T]}{P} + \frac{(1-p)E(T)^2}{P^2}$$

$$\Downarrow$$

$$E[T_i] = \frac{E[T]}{P}$$

$$V[T_i] = \frac{V[T]}{P} + \frac{(1-p)E(T)^2}{P^2}$$

Split Processes (7)

- Generalization, Cont'd

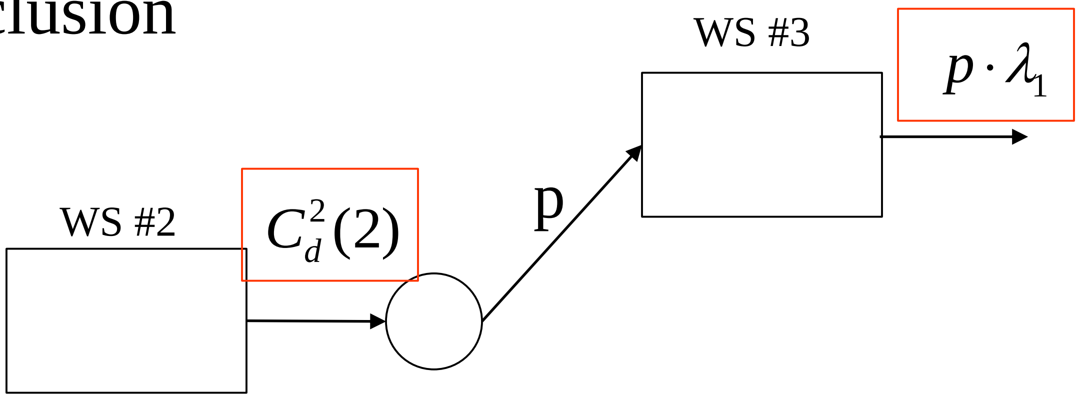
$$E[T_i] = \frac{E[T]}{P} \qquad V[T_i] = \frac{V[T]}{P} + \frac{(1-p)E(T)^2}{P^2}$$

$$C_i^2 = \frac{V[T_i]}{E[T_i]^2} = \left(\frac{V[T]}{P} + \frac{(1-p)E(T)^2}{P^2}\right) \cdot \frac{P^2}{E[T]^2}$$

$$= P \cdot C^2 + (1-P)$$

Split Processes (8)

- Conclusion



$$C_a^2(3) = p \cdot C_d^2(2) + (1 - p)$$

Homework #3

- Obtain the man flow rate